

Sampling in AdS/CFT

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Abstract

Recently, it has been proposed by Kempf a generalization of the Shannon sampling theory to the physics of curved spacetimes. With the aim of exploring the possible links between Holography and Information Theory we argue about the similitude of the reconstruction formula in the sampling theory and the bulk-to-boundary relations found in the AdS/CFT context.

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1 Introduction.

The search for possible bridges between different areas in science has been always a fertile field for researchers. Usually, the implementation of some concepts from one area into another sheds light into the features of some specific problems. Particularly, the intersection between Information Theory and Physics has inspired many people [1],[2]. In this sense Bekenstein [3] pointed out that a century of developments in physics has taught us that information is a crucial player in physical systems and processes.

Within this context Kempf [4] proposed the application of Shannon sampling theory [5] to generic curved spacetimes. His main idea is that physical fields could be constructed everywhere if sampled only at discrete points in space. These sampling points should be spaced densely enough, say of the order of the Planck distance. Recently he proved [8] the mathematical conjectures outlined in his previous papers. One of the outputs of this nice idea is that a sampling theoretic ultraviolet cutoff at the Planck scale also corresponds to a finite density of degrees of freedom for physical fields. This allows the holographic principle [9] to enter in scene. According to t' Hooft and Susskind the combination of Quantum Mechanics and Gravity requires the three dimensional world to be an image of data that can be stored on a two dimensional projection much like a holographic image. This description requires only one discrete degree of freedom per Planck area and yet it is rich enough to describe all three dimensional phenomena. This bound has not been justified but it is obvious that the assumption of these ideas imply a radical decrease in the number of degrees of freedom for describing the Universe. Maldacena's conjecture on AdS/CFT correspondence [10] is the first example realizing such a principle. Subsequently, Witten [11] proposed a precise correspondence between conformal field theory observables and those of supergravity on the AdS side.

The purpose of this note is to show, on heuristic grounds, an application of the Kempf's idea in the context of AdS/CFT correspondence. We

argue about the extreme similitude between the bulk-to-boundary formulas in AdS/CFT and the reconstruction formula proposed by Kempf.

2 Scalar amplitudes in AdS/CFT.

In this section we briefly review the essential features of bulk-to-boundary procedure for a scalar field.

The simplest way is to work in the Euclidean continuation of AdS_{d+1} which is the $Y_{-1} > 0$ sheet of the hyperboloid [11],[12]:

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^d (Y_i)^2 = -\frac{1}{a^2} \quad (1)$$

which has curvature $R = -d(d+1)a^2$. The change of coordinates:

$$z_i = \frac{Y_i}{a(Y_0 + Y_{-1})}, \quad (2)$$

$$z_0 = \frac{1}{a^2(Y_0 + Y_{-1})} \quad (3)$$

brings the induced metric to the form:

$$ds^2 = \frac{1}{a^2 z_0^2} (dz_0^2 + d\vec{z}^2). \quad (4)$$

The Euclidean action of the massive scalar field in this background is

$$S = \frac{1}{2} \int d^d z dz_0 \sqrt{g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right]. \quad (5)$$

The corresponding wave equation is:

$$\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) - m^2 \phi = 0, \quad (6)$$

$$z_0^{d+1} \frac{\partial}{\partial z_0} \left[z_0^{-d+1} \frac{\partial}{\partial z_0} \phi(z_0, \vec{z}) \right] + z_0^2 \frac{\partial}{\partial \vec{z}^2} \phi(z_0, \vec{z}) - m^2 \phi(z_0, \vec{z}) = 0. \quad (7)$$

The normalized bulk-to-boundary Green's function

$$K_\Delta(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \left(\frac{z_0}{z^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \quad (8)$$

is a solution of (7) with the necessary singular behavior as $z_0 \rightarrow 0$. Here Δ is the largest root of the characteristic equation of (7).

In the paper [11] Witten found the solution of (7) that explicitly realizes the relation between the field $\phi(z_0, \vec{z})$ in the bulk and the boundary configuration $\phi_0(\vec{x})$, that is given by

$$\phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \int d^d x \left(\frac{z_0}{z^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \phi_0(\vec{x}). \quad (9)$$

3 Sampling

In an interesting triad of papers [4],[6],[7] it was pointed out that the mathematical tools of the sampling theory could play an important role in the understanding of the space-time structure. The sampling theorem states that in order to capture a signal $f(x)$ with bandwidth ω_{max} for all x , it is sufficient to record only the signals' values at the discrete points $\{x_n\}$. These sampling points should be spaced densely enough. In other words, consider the set of square integrable functions f whose frequency content is bounded by ω_{max} . These functions are called band-limited functions. If the amplitudes $\{f(x_n)\}$ of such a function are known at equidistantly spaced discrete points $\{x_n\}$ whose spacing is π/ω_{max} then the function's amplitude $f(x)$ can be reconstructed for all x . The reconstruction formula is:

$$f(x) = \sum_{n=-\infty}^{\infty} f(x_n) \frac{\sin[(x - x_n)\omega_{max}]}{(x - x_n)\omega_{max}} \quad (10)$$

In the article [7] Kempf proposed the generalization of the sampling theory to Riemannian manifolds. There he exposed that the covariant analog of the band-limit is the cutoff of the spectrum of a scalar self-adjoint differential operator. As an explicit example he chose the Laplace-Beltrami operator Δ .

Following Kempf we start with the Hilbert space \mathcal{H} of square integrable scalar functions over the manifold. Later we consider the domain $\mathcal{D} \subset \mathcal{H}$, where the Laplacian is essentially self-adjoint. After that we define P as the

projector onto the subspace spanned by the eigenspaces of the Laplacian with eigenvalues smaller than some fixed maximum value λ_{max} . If we are working with d'Alembertians λ_{max} bounds the absolute values of the eigenvalues.

Now let us consider a physical field $|\phi\rangle$. This field belongs to the subspace $\mathcal{D}_{ph} = P.\mathcal{D}$ where are all the physical fields. It is assumed that the field's amplitudes $\phi(x_n) = (x_n|\phi)$ are known only at the discrete points $\{x_n\}$ of the manifold. While all position eigenvectors $|x\rangle$ are need to span \mathcal{H} , sufficiently dense discrete subsets $\{x_n\}$ of the set of vectors $\{P|x\rangle\}$ can span \mathcal{D}_{ph} . The field's coefficients $\{\phi(x_n)\}$ then fully determine the Hilbert space vector $|\phi\rangle \in \mathcal{D}_{ph}$ and they determine, therefore, also $(x|\phi)$ for all x . Namely, defining $K_{n\lambda} = (x_n|\lambda)$, the set of sampling points $\{x_n\}$ is sufficiently dense for reconstruction iff K is invertible. To see this, insert the resolution of the identity in terms of the eigenbasis $\{|\lambda\rangle\}$ of $-\Delta$ into $(x|\phi)$:

$$(x|\phi) = \sum_{|\lambda| < \lambda_{max}} (x|\lambda)(\lambda|\phi) d\lambda. \quad (11)$$

With K invertible one obtains $(\lambda|\phi) = \sum_n K_{\lambda,n}^{-1} \phi(x_n)$. Substituting back in (11) we obtain

$$\phi(x) = \sum_n G(x, x_n) \phi(x_n) \quad (12)$$

where

$$G(x, x_n) = \sum_{|\lambda| < \lambda_{max}} (x|\lambda) K_{\lambda,n}^{-1} d\lambda \quad (13)$$

is called the reconstruction kernel.

4 Boundary to bulk relation as a Sampling mechanism

After reading the previous sections the sharp reader may have perceived the strong resemblance between the boundary to bulk relation (9) and the reconstruction formula (12). But we think that this is not a mere formal similarity. Physically speaking the holographic principle claims that the fundamental

degrees of freedom live in the boundary. Therefore the degrees of freedom in the bulk could be seen as information “constructed” from the boundary. The holographic assumption that the boundary theory should have only a finite number of degrees of freedom per Planck area is also compatible with the sampling condition that requires the existence of a dense domain, with spacing of the order of Planck distance, in order to do the reconstruction.

In order to clarify our statement let’s consider that the function ϕ is defined at the discrete points \vec{x}_n . The particularity here lies in the fact that sampling points live in the boundary. Then the equation (9) changes to

$$\phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \sum_{\vec{x}_n} \left(\frac{z_0}{z^2 + (\vec{z} - \vec{x}_n)^2} \right)^\Delta \phi_0(\vec{x}_n). \quad (14)$$

Now, comparing this with equation (12) and treating \vec{z} as a constant we see that they agree if we take

$$G(z_0, \vec{z}, \vec{x}_n) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \left(\frac{z_0}{z^2 + (\vec{z} - \vec{x}_n)^2} \right)^\Delta. \quad (15)$$

We have assumed that the differential operator presented in (6) has a cutoff in the spectrum. We also suppose that the reconstruction stability is satisfied. Therefore the next task will be to proof (13) taking into account (15).

5 Conclusions

Potentially Information Theory could play an important role imposing some constraints to physical theories. Although this relation needs to be explored further we showed here a possible implementation of the generalization of the Shannon sampling theory to the physics of curved spacetimes, as proposed by Kempf. Seeing the sampling procedure as a mechanism that generates the bulk’s degrees of freedom seems to be a very attractive idea from the holographic point of view.

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